Bloch oscillators in a slowly perturbed external field

2008 J. Phys. A: Math. Theor. 41 265304

(http://iopscience.iop.org/1751-8121/41/26/265304)

J. Phys. A: Math. Theor. **41** (2008) 265304 (18pp) doi:10.1088/1751-8113/41/26/265304

Bloch oscillators in a slowly perturbed external field

Andrea Sacchetti

Department of Mathematics, Faculty of Sciences, Via Campi 213/B, I-41100 Modena, Italy

E-mail: Andrea.Sacchetti@unimore.it

Received 11 February 2008, in final form 14 May 2008
Published 11 June 2008
Online at stacks.iop.org/JPhysA/41/265304

Abstract
A quantum particle in a periodical lattice under the effect of an external homogeneous field shows a periodical motion, usually a named Bloch oscillator, for long times. When we introduce a weak and slowly varying inhomogeneous field then the dynamics of the quantum particle still exhibits a periodical motion but with a different period and a different width of the interval of oscillation. In this paper we obtain a formula for the dominant terms of the perturbed period and width, then we apply our result to the study of the effect of Casimir–Polder forces to a vertical Bose–Einstein condensate trapped in an optical lattice.

PACS numbers: 03.65.−w, 03.75.Lm
Mathematics Subject Classification: 81Q15, 81Vxx

1. Introduction

In this paper, we consider the motion of a quantum particle in a one-dimensional periodical lattice under the effect of an external force. It is a well-known fact that when the external force is homogeneous then the quantum particle remains confined in a finite region for a long time and finally it escapes to infinity because of the tunneling effect. In particular, such a confined motion is a periodical motion, usually named Bloch oscillators (or also Bloch oscillations), with period [3]

$$T_B = \frac{2\pi\hbar}{Fd} \tag{1}$$

where $F > 0$ is the strength of the external homogeneous force and where d is the period of the one-dimensional periodical lattice. In particular, when the quantum particle is initially prepared on one energy band (with amplitude δ) of the lattice, usually the first one, with a given mean quasi-momentum and *a quasi-momentum spread much smaller than the width of the Brillouin zone* then the region where the quantum particle remains confined for long times has a width given by (see, e.g., [23] or also the reviews in [5, 15, 20])

$$L = \frac{\delta}{F}. \tag{2}$$

Here, we explore the effect on such a periodical motion of a weak inhomogeneous force, as a perturbation of the homogeneous one. If, as we assume in this paper, the inhomogeneous external force *is a slowly varying function along the support of the wave-packet* then we expect to still observe a periodical motion for the quantum particle for long times with a slightly different period T_B^ϵ and width L^ϵ of the interval of oscillation. In particular, if we denote by $-Fx$ the potential of an external homogeneous field and by $\epsilon(x)x$ the potential of the external inhomogeneous field such that $|\epsilon(x)| \ll F$, then in this paper we prove that the shift length and the shift period depend on the mean value of the external inhomogeneous field on the unperturbed interval of oscillation. More precisely, we have that:

$$L^\epsilon = L - \frac{1}{F}\int_0^L \epsilon(x)\,\mathrm{d}x + O\left(\frac{\epsilon^2\delta}{F^3}\right) \tag{3}$$

and

$$T_B^\epsilon = T_B - \frac{2\hbar}{F^2}\int_0^{\pi/d} \epsilon\left(\frac{E_1(k) - E_1^b}{F}\right)\mathrm{d}k + O\left(\frac{\hbar\epsilon^2}{dF^3}\right) \tag{4}$$

where we assume that the state is initially prepared in the first energy band, $E_1(k)$ is the first energy band function and E_1^b is the bottom of the first energy band. Here, for the sake of simplicity, we have fixed at $x = 0$ the first endpoint of the interval $[0, L]$ of oscillation for the unperturbed particle. We remark that when $\epsilon(x)$ is simply a constant function, i.e. $\epsilon(x) \equiv \bar{\epsilon}$ for some $\bar{\epsilon}$, then (3) and (4) agree with the obvious fact that $L^\epsilon = \delta/(F + \bar{\epsilon})$ and $T_B^\epsilon = 2\pi\hbar/d(F + \bar{\epsilon})$. Higher order terms can be computed starting from the exact result (27) as done for the shift length in section 3.2.

We must underline that our results hold provided that the following two conditions are satisfied:

(a) the state is initially prepared on one energy band with a quasi-momentum spread much smaller than the Brillouin zone;
(b) the external inhomogeneous field is almost constant on the support of the wave-packet.

Hence, limit cases, as pure Bloch states or pure Wannier states, seem to be excluded because of the uncertainty principle. Indeed, pure Bloch states are sharply localized with respect to the quasi-momentum variable (in fact, they are represented by means of a Dirac delta function), but they are fully delocalized with respect to a spatial variable and so they do not meet condition (b); in contrast, Wannier states are sharply localized on one single lattice cell but they are fully delocalized with respect to the quasi-momentum variable (in fact, they are represented by means of a constant function on the Brillouin zone) and so they do not meet condition (a). However, it is possible to prepare suitable wave packets that meet both conditions (a) and (b); that is they are quite localized with respect to both spatial and quasi-momentum variables and they exhibit a soliton-like motion where the centroid of the wave packet periodically moves along the interval of oscillation with width L maintaining a well-localized shape.

As an application of equations (3) and (4) we consider the recent experiment proposed by Carusotto *et al* [7] where they explore the effect of Casimir–Polder forces on atomic Bloch oscillators (in this case the external homogeneous force is the gravitational force and the lattice is aligned along the vertical direction). In fact, Bose–Einstein condensates are an ideal tool for the study of quantum effects connected with the dynamics of a single particle in a periodical lattice. Indeed, the condensate is represented by means of only one wavefunction since all the neutral atoms occupy the same quantum state. The periodical lattice potential is then realized by means of an optical lattice which does not have defects and its strength and period can be easily tuned (for experimental observations of Bloch oscillators in optical lattices see, e.g., the papers [9, 16, 21]). In such a periodic potential the atomic cloud is naturally described by means of a Bloch oscillator and the condensate wavefunction can be considered a sharply localized wavefunction in the quasi-momentum space (see, e.g., section 5.2.3 in [8]).

In particular, in the paper [7] they consider a BEC with 2×10^{4} ^{40}K atoms prepared in an optical lattice with wavelength $\lambda = 873$ nm, that is the lattice period is $d = \lambda/2 = 436.5$ nm, and a vertical extension L of 4 μm, that is around eight lattice cells. In their paper they consider a model where the external Casimir–Polder force, due to a sapphire surface at distance D from the BEC, where D goes from 4 μm to 10 μm, is assumed, at first, spatially homogeneous along the atomic cloud. Then they numerically investigate the effect of the spatial inhomogeneity of the Casimir–Polder force along the atomic cloud proving that the relative shift in the Bloch oscillators period due to the Casimir–Polder force is actually different with respect to the previous case. In this paper, we compute the relative shift by making use of formula (4) and our result qualitatively agrees with [7]; in fact, the relative shift is different with respect to the case of the homogeneity of the Casimir–Polder force and such a difference is similar to that numerically obtained by [7].

The paper is organized as follows.

In section 2 we collect some fundamental results on Bloch oscillators, previously obtained by Grecchi and Sacchetti [10–12]. In particular, section 2.1 is devoted to the study of the motion of Bloch oscillators in the quasi-momentum representation (also called crystal momentum representation), the acceleration theorem is stated in its classical version (10) and, furthermore, in the version (12) where a new phase term (13) has been introduced by Grecchi and Sacchetti [10]; section 2.2 is devoted to the computation of the centroid (17) and the variance (19) of Bloch oscillators in the spatial variable; in section 2.3 different initial wave packets are considered and an expression of an optimal wave packet (23) which translates in space maintaining its shape is given. The proofs of some of these results are collected in the appendix.

In section 3, we considered the case of Bloch oscillators in a slowly varying external field in the general case, in section 3.1, and then in section 3.2, for wave packets sharply localized in the quasi-momentum space. In particular, our main results (3) and (4) are obtained.

In section 4, we consider the model introduced by Carusotto *et al* [7] and we compute the period and length shift due to the Casimir–Polder force by means of our theoretical results (3) and (4).

2. Bloch oscillators

The wave packet $\psi(x,t)$ of a quantum particle in a one-dimensional periodical lattice under an external homogeneous force along the direction of the lattice satisfies the time-dependent Schrödinger equation

$$\begin{cases} i\hbar \dfrac{\partial}{\partial t}\psi = H_F\psi, & \psi = \psi(x,t) \in L^2(R, \mathrm{d}x) \\ \psi(x,0) = \psi^0(x) \end{cases} \tag{5}$$

where

$$H_F = H_{\mathrm{Bloch}} - Fx, \qquad H_{\mathrm{Bloch}} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x).$$

In the usual setting of solid-state physics m is the effective mass of the quantum particle, F is the constant strength of the external force along the direction x of the one dimensional crystal and $V(x) = V(x+d)$ is the crystal potential with period d. H_{Bloch} is the self-adjoint Bloch operator defined on the Hilbert space $L^2(R, \mathrm{d}x)$ whose spectrum is absolutely continuous and it is given by bands

$$\sigma(H_{\text{Bloch}}) = \cup_{n=1}^{\infty}\left[E_n^b, E_n^t\right], \qquad E_n^t \leqslant E_{n+1}^b < E_{n+1}^t.$$

We assume here that the wave packet at the initial time $\psi^0(x)$ satisfies the normalization condition $\int_{-\infty}^{+\infty} |\psi^0(x)|^2 \, \mathrm{d}x = 1$.

In fact, when the state is initially prepared on one band, e.g. the first one, then, because of the periodicity of the crystal potential, we expect to observe that the dominant part of the wave packet performs a periodic motion (Bloch oscillator) with period T_B given by (1) and this periodic motion occurs within an interval of width L given by (2), where $\delta = E_1^t - E_1^b$ is the width of the first energy band.

Since T_B is the actual unit of time it is more convenient to rescale the time as

$$\tau = \frac{Ft}{\hbar}$$

so that the Bloch period is given by

$$\tau_B = \frac{2\pi}{d}$$

and the time-dependent Schrödinger equation takes the form

$$\mathrm{i}F\frac{\partial}{\partial\tau}\psi = H_F\psi, \qquad \psi = \psi(x,\tau). \tag{6}$$

2.1. *Motion of the wave packet in the quasi-momentum representation*

The quasi-momentum (also called crystal momentum) representation of the wave packet is employed, which means that the electron wave packet $\psi(x,\tau)$ is expanded in Bloch functions

$$\psi(x,\tau) = \sum_{n=1}^{\infty}\int_{\mathcal{B}} \mathrm{a}_n(k,\tau)\varphi_n(k,x)\,\mathrm{d}k, \tag{7}$$

where φ_n are the Bloch functions, k is the quasi-momentum variable belonging to the Brillouin zone $\mathcal{B} = [0, 2\pi/d)$. Since the Bloch functions φ_n are normalized in such a way

$$\int_{-\infty}^{+\infty} \bar{\varphi}_n(k,x)\varphi_m(k',x)\,\mathrm{d}x = \delta_n^m\delta(k-k')$$

the functions $a(k,\tau) = (a_n(k,\tau))_{n=1}^{\infty}$ representing the wave packet in the quasi-momentum representation are periodic functions with respect to the quasi-momentum variable k, $a_n(k,\tau) = a_n(k+2\pi/d,\tau)$, and they are defined as

$$a_n(k,\tau) = \int_{-\infty}^{+\infty} \psi(x,\tau)\bar{\varphi}_n(k,x)\,\mathrm{d}x, \qquad n = 1, 2, \ldots.$$

The normalization of the wave packet $\psi(x,\tau)$ implies that

$$\sum_{n=1}^{\infty}\int_{\mathcal{B}} |a_n(k,\tau)|^2\,\mathrm{d}k = \int_{-\infty}^{+\infty} |\psi(x,\tau)|^2\,\mathrm{d}x = \int_{-\infty}^{+\infty} |\psi^0(x)|^2\,\mathrm{d}x = 1$$

In such a representation equation (6) takes the form (see equation (6.1.8) in [5])

$$\left[E_n(k) - \mathrm{i}F\frac{\partial}{\partial k} - \mathrm{i}F\frac{\partial}{\partial\tau}\right]a_n(k,\tau) - F\sum_{\ell=1}^{\infty}X_{n,\ell}(k)a_\ell(k,\tau) = 0 \tag{8}$$

with initial condition

$$a_n^0(k) = a_n(k,0) = \int_{-\infty}^{+\infty} \psi^0(x)\bar{\varphi}_n(k,x)\,\mathrm{d}x.$$

The band functions $E_n(k)$ are periodic functions with period $2\pi/d$, they are even functions, i.e. $E_n(-k) = E_n(k)$, such that

$$E_n^b = \begin{cases} E_n(0), & n \text{ odd} \\ E_n(\pi/d), & n \text{ even} \end{cases} \qquad \text{and} \qquad E_n^t = \begin{cases} E_n(\pi/d), & n \text{ odd} \\ E_n(0), & n \text{ even.} \end{cases}$$

The coupling terms $X_{n,\ell}(k)$ are given by

$$\overline{X_{\ell,n}}(k) = X_{n,\ell}(k) = \mathrm{i}\frac{2\pi}{d}\int_0^d \bar{u}_n(k,x)\frac{\partial u_\ell(k,x)}{\partial k}\,\mathrm{d}x$$

where

$$\varphi_n(k,x) = \mathrm{e}^{\mathrm{i}kx}u_n(k,x), \qquad u_n(k,x+d) = u_n(k,x). \tag{9}$$

Remark 1. By means of a suitable gauge of the phase term of the Bloch functions φ_n (see, e.g., [22]) the *intra-band* coupling term $X_{n,n}(k)$ can be choosen to be a constant term; in particular, in the case of symmetric periodic potential, it can be choosen to be exactly zero (see, e.g., [2]).

Following Callaway [5] we obtain the motion of the centroid of the wave packet in the quasi-momentum representation. Let $\langle k\rangle^\tau$ be the expectation value of the quasi-momentum variable k on the state a defined as

$$\langle k\rangle^\tau = \langle a(k,\tau)|k|a(k,\tau)\rangle = \sum_{n=1}^{\infty}\int_{\mathcal{B}} k|a_n(k,\tau)|^2\,\mathrm{d}k.$$

Then

$$\langle k\rangle^\tau = \langle k\rangle^0 + \tau \tag{10}$$

where $\langle k\rangle^0$ is the expectation value of the quasi-momentum variable in the initial state.

Equation (10) is often called the *acceleration theorem*.

As a comment to equation (10) Callaway (see [5], p 468) pointed out that: *'One often finds in the literature the alternative form* $\frac{\mathrm{d}k}{\mathrm{d}\tau} = 1$ *which must be understood as referring to the centroid of the packet'*. In fact, in many solid-state textbooks this result is usually applied to a pure Bloch state where (see, e.g., [15], p 191): *'an electron which stays in a given state k will appear to change its properties in terms of the states classified in k at* $\tau = 0$ *as if* $\frac{\mathrm{d}k}{\mathrm{d}\tau} = 1$. *That is, an electron in a pure Bloch state at* $\tau = 0$ *will at a later time* τ *be in a state having the original k, but with all the other properties of the state originally at* $k - \tau$*.'* According to the criticism by Bouchard and Luban (see appendix 3 in [4]), if the initial state is not a pure Bloch state then this last argument does not apply.

In order to overcome this flaw we state the following result which extends such an analysis to any initial state. To this end, we consider the behavior of Bloch oscillators in the *decoupled band approximation* obtained by neglecting the inter-band interaction and where equation (8) takes the form

$$\left[E_n(k) - FX_{n,n}(k) - \mathrm{i}F\frac{\partial}{\partial k} - \mathrm{i}F\frac{\partial}{\partial\tau}\right]a_n(k,\tau) = 0, \qquad n = 1,2,\ldots. \tag{11}$$

The solution $a_n(k, \tau)$ of the decoupled band approximation (11) satisfying the initial condition $a_n(k, 0) = a_n^0(k)$ is given by [11, 12]

$$a_n(k, \tau) = e^{i\theta_n(k,\tau)} a_n^0(k - \tau) \tag{12}$$

where

$$\theta_n(k, \tau) = -\frac{1}{F} \int_{k-\tau}^{k} [E_n(q) - F X_{n,n}(q)]\, dq \tag{13}$$

is a phase factor.

Remark 2. From (12) it follows that the time behavior of Bloch oscillators in the quasi-momentum representation is given by means of a uniform translation $k \to k - \tau$ (as usually expected) together with a change in phase. The term $\frac{1}{F}\int_{k-\tau}^{k} E_n(q)\, dq$ of the phase factor is, for what known by us, a completely new term and it plays a crucial role in order to understand the dynamics of the wave packet $\psi(x, \tau)$. Keeping in mind (remark 1) that the intra-band term $X_{n,n}$ is a constant function then the term $X_{n,n}\tau$ in the phase factor is independent of k; this term has been obtained by Zak and it is related to the fact that Wannier–Stark ladders are translationally invariant: the position of the discrete levels (with respect to the band energy) must be independent of the choice of the origin in the crystal cell [17, 18].

Remark 3. We point out that the tunneling effect between the bands is not considered in this approximation because we neglect the coupling terms $X_{n,\ell}$ with $n \neq \ell$; actually, this fact is not really crucial since the tunneling time between bands is, typically, much larger than the period of Bloch oscillators. For a complete treatment where the tunneling effect is considered we refer to [10, 11].

2.2. *Expectation value of the position operator*

Going back to the position representation, from equations (7) and (12) it follows that the solution of the time-dependent Schrödinger equation (6) has the dominant term given by

$$\psi(x, \tau) = \sum_{n=1}^{\infty} \int_{\mathcal{B}} a_n^0(k - \tau) e^{i\theta_n(k,\tau)} \varphi_n(k, x)\, dk \tag{14}$$

at least for times smaller than the tunneling time.

If the initial state $\psi^0(x)$ is a *pure Bloch state* prepared on one single band (e.g. the first one), that is $\psi^0(x) = \varphi_1(x, k_0)$ for some k_0, then the solution $a_n(k, \tau)$ in (12) takes the form $a_n(k, \tau) = \delta_n^1 e^{i\theta_n(k,\tau)} \delta(k - \tau - k_0)$ and the wave packet takes the form of the *Houston function* (see, e.g., equation (6.1.41) in [5])

$$\psi(x, \tau) = \psi^0(x, k_0 + \tau) e^{i\theta_1(k_0+\tau,\tau)} = \varphi_1(k_0 + \tau, x) e^{-\frac{i}{F}\int_{k_0}^{k_0+\tau} E_1(q)\, dq}\, e^{iX_{1,1}\tau}$$

as proved by Houston [13] and Wannier (see equation (45) in [20]). We emphasize that, in such a case, the phase factor (13) does not play any particular role.

In contrast, if the initial state *does not coincide with a pure Bloch state*, the phase factor (13) appearing in (12) plays a crucial role in order to obtain the dynamics of the wave packet $\psi(x, \tau)$ in the position representation. Here, from (14), we give the expectation value and the variance of the position operator for any initial state $\psi^0(x)$.

Let

$$\langle x\rangle^\tau = \langle \psi(x,\tau)|x|\psi(x,\tau)\rangle = \int_{-\infty}^{+\infty} x|\psi(x,\tau)|^2\,\mathrm{d}x$$

be the expectation value of the position observable and let $S^\tau = \langle [x - \langle x\rangle^\tau]^2\rangle^\tau$ be its variance. Then, in the limit of a small external force and for times of the order of the Bloch period, the dominant term of the expectation value and the dominant term of the variance of the position operator are given by

$$\langle x\rangle^\tau - \langle x\rangle^0 = \sum_{n=1}^{+\infty} \frac{1}{F}\int_{\mathcal{B}} \left|a_n^0(k)\right|^2 [E_n(k+\tau) - E_n(k)]\,\mathrm{d}k \tag{15}$$

and, if a^0 is a real-valued function, then

$$S^\tau - S^0 = -([\langle x\rangle^\tau]^2 - [\langle x\rangle^0]^2) + \frac{1}{F^2}\sum_{n=1}^{\infty}\int_{\mathcal{B}} [E_n(k+\tau) - E_n(k)]^2 \left|a_n^0(k)\right|^2\,\mathrm{d}k. \tag{16}$$

Remark 4. This result extends that obtained by Bouchard and Louban (see equation (16) in [4]) for the special case of a state initially prepared on the first band, where they assumed that the initial state a_1^0 is independent of the quasi-momentum k and where the first band is given by a cosine function.

Equation (15) can be also written as

$$\langle x\rangle^\tau - \langle x\rangle^0 = \frac{1}{F}[\langle E(k+\tau)\rangle^0 - \langle E(k)\rangle^0] \tag{17}$$

where on the left-hand side the mean value has to be intended on the position variable while in the right-hand side the mean value has to be intended in the quasi-momentum variable, i.e. ,

$$\langle E(k)\rangle^\tau = \langle a(k,\tau)|E(k)|a(k,\tau)\rangle = \sum_{n=1}^{+\infty}\int_{\mathcal{B}} |a_n(k,\tau)|^2 E_n(k)\,\mathrm{d}k.$$

In particular, it follows that

$$\langle x\rangle^{\tau+\mathrm{d}\tau} - \langle x\rangle^\tau = \frac{\langle E(k+\tau+\mathrm{d}\tau)\rangle^0 - \langle E(k+\tau)\rangle^0}{F} \tag{18}$$

and

$$\frac{\mathrm{d}\langle x\rangle^\tau}{\mathrm{d}\tau} = \frac{1}{F}\langle E'(k+\tau)\rangle^0 \qquad \text{and} \qquad \frac{\mathrm{d}^2\langle x\rangle^\tau}{\mathrm{d}\tau^2} = \frac{1}{F}\langle E''(k+\tau)\rangle^0$$

where $' = \frac{\partial}{\partial k}$ denotes the derivative with respect to the quasi-momentum variable.

For what concerns the variance we remark that if a^0 is a real-valued function then the function ψ^0 has the property of symmetry $\psi^0(-x) = \overline{\psi^0}(x)$; thus $\langle x\rangle^0 = 0$ and equation (16) can be written as

$$S^\tau - S^0 = \frac{1}{F^2}[\langle [E(k+\tau) - E(k)]^2\rangle^0 - [\langle E(k+\tau) - E(k)\rangle^0]^2] \tag{19}$$

where

$$S^0 = \sum_{n=1}^{\infty}\int_{\mathcal{B}} \left|\frac{\partial a_n^0(k)}{\partial k}\right|^2 \mathrm{d}k. \tag{20}$$

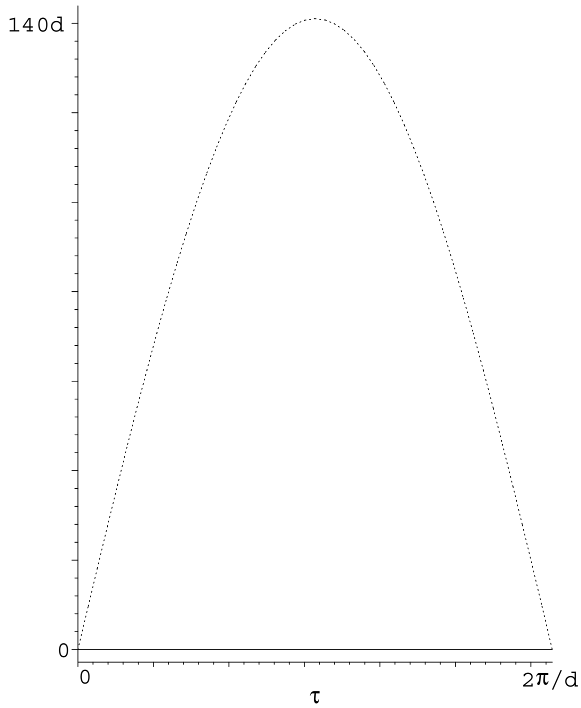

Figure 1. We plot the function $\langle x\rangle^\tau$ (full line), which represents the motion of the centroid of the wave packet, and the standard deviation σ^τ (dotted line) for a state initially prepared on a Wannier state. In such a case we do not have the motion of the centroid. Hence, the Bloch oscillator performs an actual breathing motion, that is the wave packet, initially localized on a single site of the lattice, periodically enlarges and shrinks without moving its center. Here, for the sake argument, we choose $\delta/F = 200d$.

2.3. Periodic motion of the wave packet in the position representation: breathing motion versus soliton-like shape

In order to better understand the motion of Bloch oscillators initially prepared on the first band we compute the centroid $\langle x\rangle^\tau$ and the standard deviation $\sigma^\tau = \sqrt{S^\tau}$ of the wave packet for different given initial states. Here, for the sake of definiteness, we assume that the first band function is simply given by $E_1(k) = \frac{1}{2}\delta\,[1-\cos(kd)]$, where δ is the width of the first band and F is chosen such that $\delta/F = 200d$; again, d is the period of the crystal.

In the first case we consider a state that initially coincides with an exact Wannier state where $a_1^0(k) = \sqrt{d/2\pi}$, that is the electron wave packet is initially localized on one site of the lattice. In such a case the wave packet exhibits a symmetrical motion and the centroid remains fixed; in contrast, its variance is given by $S^\tau = \frac{1}{2}\frac{\delta^2}{F^2}\sin^2\left(\frac{\tau d}{2}\right)$. Hence, the Bloch oscillator performs an actual breathing motion, that is the wave packet, initially localized on a single site of the lattice, periodically enlarges and shrinks without moving its center (see figure 1). As a result, we expect to have no electronic current, according to the fact that there is no conductance in full bands [19].

In contrast, in the case of a pure Bloch state we then have the opposite situation for what concerns the motion of the centroid of the wave packet (see figure 2); that is the centroid of

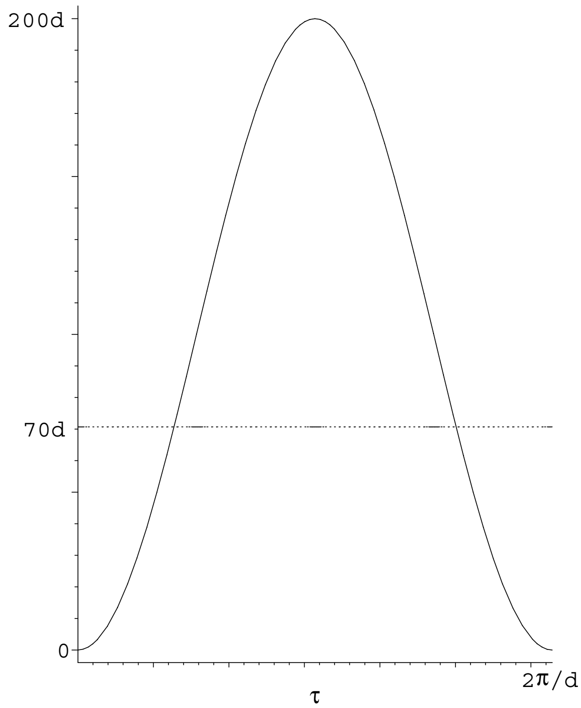

Figure 2. We plot the function $\langle x\rangle^\tau$ (full line), which represents the motion of the centroid of the wave packet, and the standard deviation σ^τ (dots line) for a state initially prepared on a pure Bloch state, that is the initial state is given by (21) with $\rho = 0.01$. In such a case the centroid of the wave packet periodically moves in an interval of width $\delta/F = 200d$. The variance is almost constant but it is very large, that is the wave packet is practically localized in an interval with width of the order of the width of the interval of oscillation. In fact in the limiting case of an exact pure Bloch state, i.e. $\rho = 0$, then the wave packet is fully delocalized in space.

the wave packet performs a periodical motion within the interval of width δ/F, but the state is largely delocalized within such an interval.

We now consider an optimal situation, where the state performs a full periodical motion on an interval of width of order δ/F, as in the case of a pure Bloch state, but the shape remains well localized. To this end, we consider that a state is initially localized in the quasi-momentum variable around $k_0 = 0$ according to the following distribution:

$$a_1^0(k) = c\mathrm{e}^{-k^2/2\rho^2}, \qquad c = \left[\sqrt{\pi}\rho\,\mathrm{erf}\left(\frac{\pi}{\rho d}\right)\right]^{-1/2} \tag{21}$$

for $\rho > 0$, and periodically arranged on the Brillouin zone. The parameter ρ will be chosen later such that $\rho d \ll 1$ in order to optimize the variance S^τ of the wave packet.

A simple calculation gives that

$$\langle x\rangle^\tau - \langle x\rangle^0 = \delta C_1(d\rho)\sin^2(\tau d/2),$$

where

$$C_1(\xi) = \frac{\mathrm{Re}\left[\mathrm{erf}\left(\frac{\pi}{\xi} - \frac{1}{2}\mathrm{i}\xi\right)\right]}{\mathrm{erf}\left(\frac{\pi}{\xi}\right)}\mathrm{e}^{-\xi^2/4},$$

and

$$S^{\tau} = \frac{1}{2\rho^2}C_0(d\rho) + \frac{\delta^2}{2F^2}\sin^2\left(\frac{\tau d}{2}\right)[1-\cos(\tau d)C_2(d\rho)] - \frac{\delta^2}{F^2}C_1^2(d\rho)\sin^4(\tau d/2)$$
$$\approx \frac{1}{2\rho^2} + \frac{(\rho d)^2}{8}\frac{\delta^2}{F^2}\sin^2(\tau d) \leqslant \frac{1}{2\rho^2} + \frac{(\rho d)^2}{8}\frac{\delta^2}{F^2}$$

in the limit of small ρd, where

$$C_0(\xi) := 1 - \frac{2\sqrt{\pi}\,\mathrm{e}^{-\pi^2/\xi^2}}{\xi\,\mathrm{erf}(\pi/\xi)}$$

and

$$C_2(\xi) := \frac{\mathrm{Re}\left[\mathrm{erf}\left(\frac{\pi}{\xi} - \mathrm{i}\xi\right)\right]}{\mathrm{erf}\left(\frac{\pi}{\xi}\right)}\mathrm{e}^{-\xi^2}.$$

We recall that $\mathrm{erf}(x) \approx 1$ as x goes to $+\infty$.

Hence, it is convenient to choose $\rho = \sqrt{2F/\delta d}$ in order to minimize S^{τ}:

$$S^{\tau} \leqslant 2S^0 = \frac{1}{\rho^2} = \frac{\delta d}{2F}.$$

With this choice of $\rho = \sqrt{2F/\delta d}$ then condition $\rho d \ll 1$ is implied by $d \ll \frac{\delta}{F}$.

Therefore we can conclude that: if the parameters are such that

$$d \ll \frac{\delta}{F} \tag{22}$$

then the motion of the optimal wave packet

$$a_1^0(k) = \frac{1}{\sqrt[4]{2\pi F/\delta d}}\mathrm{e}^{-4Fk^2/\delta d} \tag{23}$$

is similar to that of a soliton (that is, roughly speaking, the wave packet translates in space maintaining its initial shape) whose support is contained in an interval with width of the order $\sqrt{\delta d/F}$. In particular, the centroid of the wave packet moves forward for a length

$$L = C_1(d\rho)\frac{\delta}{F} = \left[1 - \frac{1}{2}(d\rho)^2 + \cdots\right]\frac{\delta}{F} \approx \frac{\delta}{F} - d \tag{24}$$

maintaining a well-localized shape and then it goes back to the initial position (see figure 3).

3. Bloch oscillators in a slowly varying external field

3.1. Theoretical formula in the general framework

We now consider the case when the external field is no more constant, but it slowly depends on the position; hence, its perturbed potential can be written as $\epsilon(x)x$ for some slowly varying function $\epsilon(x)$. The time-dependent Schrödinger equation takes the form

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi - [F+\epsilon(x)]x\psi$$

with the initial condition $\psi^0(x) = \psi(x,0)$. In fact, $\epsilon(x)$ is a small and slowly varying perturbation, that is $|\sup_x \epsilon(x)| \ll |F|$ in the considered domain and $\epsilon(x)$ is assumed *to be almost constant on the support of the wave packet* $\psi(x,t)$. Under this condition we heuristically expect that the motion of the wave packet $\psi(x,t)$ is similar to the motion of the

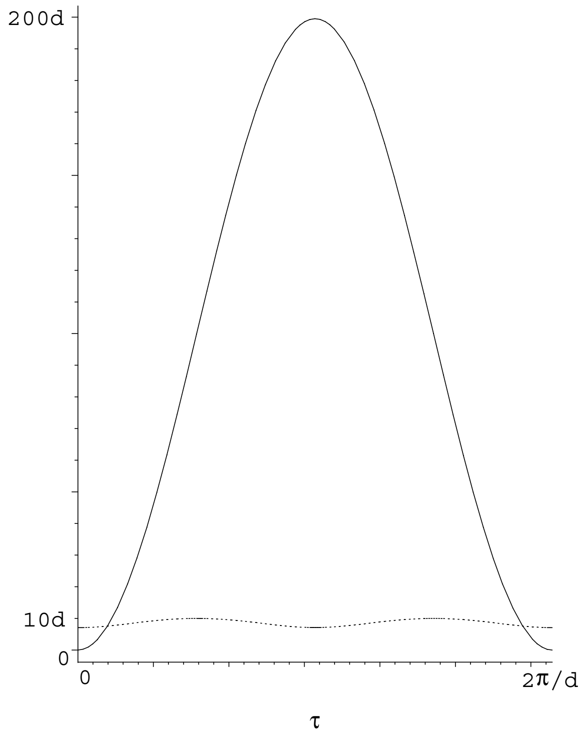

Figure 3. We plot the function $\langle x\rangle^\tau$ (full line), which represents the motion of the centroid of the wave packet, and the standard deviation σ^τ (dotted line) for a state initially prepared on an optimal wave packet (23). In such a case the centroid of the wave packet periodically moves in an interval of width $\delta/F - d$. The variance is almost constant and it is quite small, that is the wave packet is well localized and it has a soliton-like motion.

unperturbed Bloch oscillator, i.e. it is still a periodic motion, with a period T_B^ϵ depending on ϵ, within an interval of the width L^ϵ depending on ϵ. In order to compute this period and the width of the interval of oscillation we split this interval in a sequence of points x_i, $i = 0, 1, 2, \ldots, n$, such that $x_{i+1} - x_i = \frac{L^\epsilon}{n}$, n will be chosen large enough; we denote $F_i = F + \epsilon(x_i)$ and t_i the instant when the centroid of the particle is at the position x_i, that is $\langle x\rangle^{t_i} = x_i$. For the sake of definiteness we assume that $t_0 = 0$, $\langle x\rangle^{t_0} = x_0 = 0$ and $\langle x\rangle^{t_n} = L^\epsilon$. Furthermore, we denote by τ^ϵ the rescaled time, depending on ϵ, such that $\tau_0^\epsilon = t_0 = 0$ and

$$\mathrm{d}\tau_i^\epsilon = \tau_{i+1}^\epsilon - \tau_i^\epsilon = \frac{F_{i+1}}{\hbar}(t_{i+1} - t_i), \qquad i = 0, 1, 2, \ldots, n. \tag{25}$$

Since we have assumed that the external perturbed potential is almost constant on the support of the wave packet then, for n large enough, from (18) we can write that

$$\begin{aligned}\mathrm{d}x_i &:= \langle x\rangle^{t_{i+1}} - \langle x\rangle^{t_i} \\ &= \frac{1}{F_i}\left[\left\langle E\left(k + \mathrm{d}\tau_i^\epsilon + \tau_i^\epsilon\right)\right\rangle^0 - \left\langle E\left(k + \tau_i^\epsilon\right)\right\rangle^0\right] \\ &= \frac{1}{F_i}\left\langle E'\left(k + \tau_i^\epsilon\right)\right\rangle^0 \mathrm{d}\tau_i^\epsilon + \frac{1}{F_i} O\left(\mathrm{d}\tau_i^{\epsilon\,2}\right)\end{aligned}$$

Thus, we obtain the following relations:

$$[F + \epsilon(\langle x\rangle^{\tau^\epsilon})]\frac{\mathrm{d}x}{\mathrm{d}\tau^\epsilon} = \langle E'(k+\tau^\epsilon)\rangle^0 \tag{26}$$

and

$$\int_0^{L^\epsilon} [F + \epsilon(x)]\,\mathrm{d}x = \int_0^{\tau_p^\epsilon} \langle E'(k+\tau^\epsilon)\rangle^0\,\mathrm{d}\tau^\epsilon. \tag{27}$$

where τ_p^ϵ is the (rescaled) time for which the centroid of the particle moves from the initial position $x_0 = 0$ to the endpoint $x = L^\epsilon$ of the interval of oscillation. Hence, the period of oscillation, which corresponds to the twice of τ_p^ϵ, *will depend on the mean value of the external force* $\epsilon(x)$ *along the width of oscillation*.

3.2. Period of oscillation for a well-localized wave packet

In order to apply this formula we consider, at first, the case of an initial wave packet ψ^0 given by a *pure Bloch state* prepared on the first band: that is $a_n^0(k) = \delta_1^n\delta(k)$. In such a case

$$\langle E'(k+\tau^\epsilon)\rangle^0 = E_1'(\tau^\epsilon)$$

and the instant τ_p^ϵ when the motion of the centroid inverts its velocity is given by $\tau_p^\epsilon = \frac{\pi}{d}$. Hence, (27) takes the form

$$\int_0^{L^\epsilon} [F + \epsilon(x)]\,\mathrm{d}x = \int_0^{\tau_p^\epsilon} E_1'(\tau^\epsilon)\,\mathrm{d}\tau^\epsilon = \delta$$

where $\delta = E_1^t - E_1^b$ is the width of the first band, $E_1^b = E_1(0)$ and $E_1^t = E_1(\pi/d)$ are the bottom and the top of the first band. If we set $L^\epsilon = L + r_\epsilon$, where $L = \delta/F$ is the width of the interval of oscillation of Bloch oscillators initially prepared on a pure Bloch state, then the above equation becomes

$$\int_0^L F\,\mathrm{d}x + \int_L^{L+r_\epsilon} F\,\mathrm{d}x + \int_0^{L+r_\epsilon} \epsilon(x)\,\mathrm{d}x = \delta. \tag{28}$$

Thus, we have the following equation for r_ϵ:

$$Fr_\epsilon + \int_0^L \epsilon(x)\,\mathrm{d}x + \int_L^{L+r_\epsilon} \epsilon(x)\,\mathrm{d}x = 0;$$

that is r_ϵ is the solution of the fixed point equation $R(r) = r$ where

$$R(r) = -\frac{1}{F}\left[\int_0^L \epsilon(x)\,\mathrm{d}x + \int_L^{L+r}\epsilon(x)\,\mathrm{d}x\right].$$

The solution of such an equation is given by means of the iterative procedure $r_{n+1} = R(r_n)$ where $r_0 = 0$. So, at the first-order approximation

$$r_1 = R(r_0) = -\frac{1}{F}\int_0^L \epsilon(x)\,\mathrm{d}x,$$

The second-order approximation yields - (3), indeed

$$r_2 = R(r_1) = -\frac{1}{F}\int_0^L \epsilon(x)\,\mathrm{d}x - \frac{1}{F}\int_L^{L+r_1}\epsilon(x)\,\mathrm{d}x = -\frac{1}{F}\int_0^L \epsilon(x)\,\mathrm{d}x + O\left(\frac{\epsilon^2\delta}{F^3}\right).$$

For what concerns the period T_B^ϵ from (25) we can write

$$\begin{aligned}\frac{1}{2}T_B^\epsilon &= \int_0^{T_B^\epsilon/2}\mathrm{d}t = \int_0^{\pi/d}\frac{\hbar}{F+\epsilon(\langle x\rangle^{\tau^\epsilon})}\,\mathrm{d}\tau^\epsilon \\ &= \hbar\int_0^{L^\epsilon}\frac{1}{E_1'\left[E_1^{-1}\left(Fx+\epsilon(x)x+E_1^b\right)\right]}\,\mathrm{d}x\end{aligned} \tag{29}$$

since

$$\mathrm{d}\tau^\epsilon = \frac{F+\epsilon(x)}{E_1'\left[E_1^{-1}\left(Fx+\epsilon(x)x+E_1^b\right)\right]}\,\mathrm{d}x.$$

On the other hand, it is also possible to directly compute the dominant term of T_B^ϵ obtaining that

$$\begin{aligned}\frac{1}{2}T_B^\epsilon &= \int_0^{\pi/d}\frac{\hbar}{F+\epsilon(\langle x\rangle^{\tau^\epsilon})}\,\mathrm{d}\tau^\epsilon \\ &= \frac{\hbar}{F}\frac{\pi}{d} - \frac{\hbar}{F^2}\int_0^{\pi/d}\epsilon(\langle x\rangle^{\tau^\epsilon})\,\mathrm{d}\tau^\epsilon + O\left(\frac{\hbar\epsilon^2}{F^3 d}\right)\end{aligned} \tag{30}$$

since $\frac{1}{1+x} = 1 - x + (x^2)$ when $|x| \ll 1$. Hence,

$$\begin{aligned}T_B^\epsilon &\approx \frac{\hbar}{F}\frac{2\pi}{d} - \frac{2\hbar}{F}\int_0^{L^\epsilon}\frac{\epsilon(x)}{E_1'\left[E_1^{-1}\left(Fx+\epsilon(x)x+E_1^b\right)\right]}\,\mathrm{d}x \\ &\approx \frac{\hbar}{F}\frac{2\pi}{d} - \frac{2\hbar}{F}\int_0^{L}\frac{\epsilon(x)}{E_1'\left[E_1^{-1}\left(Fx+E_1^b\right)\right]}\,\mathrm{d}x \\ &\approx T_B - \frac{2\hbar}{F^2}\int_0^{\pi/d}\epsilon\left(\frac{E_1(k)-E_1^b}{F}\right)\mathrm{d}k\end{aligned} \tag{31}$$

so obtaining (4).

Remark 5. We underline that when $\epsilon \equiv \bar{\epsilon}$ is a constant term then the above results (28) and (29) simply reduce to the well-known result:

$$L^\epsilon = \frac{\delta}{F+\bar{\epsilon}} \qquad \text{and} \qquad T_B^\epsilon = \frac{2\pi\hbar}{d(F+\bar{\epsilon})}.$$

We should underline that the above arguments hold provided that the external field $\epsilon(x)$ slowly changes along the support of the wave packet. Unfortunately, pure Bloch states are quasiperiodic functions with respect to the spatial variable and their support is not finite. However, under condition (22) we can prepare the initial state in order to have an optimal wave packet (23) which exhibits a soliton-type shape with a well-localized support. In such a case if the external field is slowly varying in intervals of width of the order $\sqrt{S^\tau} \approx \sqrt{\frac{\delta d}{F}}$, then the above procedure can be applied and from (26), (27) and (31) we obtain the width L^ϵ of oscillation and the period T_B^ϵ where L is now given by (24).

4. Measurement of Casimir–Polder forces using Bloch oscillators

We now consider the effect of Casimir–Polder forces between quantum particles and the dielectric surface as in the explicit model considered in [7]. The physical system they consider consists of a sample of ultracold fermionic atoms trapped in a one-dimensional optical lattice aligned along the vertical axis. In fact, the dynamics of Bose–Einstein condensed states is described by means of nonlinear Gross–Pitaevskii equations. However, when the wave packet is sharply localized and the strength of the nonlinear term of the Gross–Pitaevskii equation is of the same order of the external field then it is expected that the time dynamics of the full wave packet is not affected by the non linear term, up to a phase term (see, e.g., [6], see also the analysis by [1]). Because of the constant gravity acceleration the atoms, initially cooled in a trapping potential, start to perform Bloch oscillations in the periodical lattice when the trapping potential is switched off. If a surface is close to the lattice then additional forces between the atoms and the surface occur and the Bloch oscillations will be affected. In particular, the Bloch period and the length of the interval of oscillation will be affected; hereafter, according to the notation introduced by [7], we will denote by ΔT_B and ΔL the shift period and the shift length.

Hence, Bloch oscillators realize a powerful sensor for the detection of surface forces.

In particular, in [7] they consider, as additional forces between the atoms and the surface, the Casimir–Polder force $F_{\rm CP}(x)$ described by means of the following equation for the energy potential:

$$V_{\rm CP}(x) = -\frac{k_B T \alpha_0}{4x^3}\frac{\epsilon_0 - 1}{\epsilon_0 + 1} \tag{32}$$

where α_0 is the static atomic polarizability, ϵ_0 is the static dielectric constant of the material composing the surface (which is placed at $x = 0$, with the x-axis being oriented downwards). The external homogeneous force F is simply given by mg where m is the mass of the particle and g is the gravity acceleration. The interval of oscillation has length $L = 4\ \mu$m, the distance D from the surface to the center of the atomic cloud goes from 4 to 10 μm and the period d of the lattice is equal to $d = \frac{1}{2}\lambda$, where $\lambda = 873$ nm.

If we assume, at first, that the external Casimir–Polder force (32) acts on the atomic cloud as an homogeneous force then the period shift is simply given by formula (4) in [7], that is

$$\frac{\Delta T_B}{T_B} \approx -\frac{F_{\rm CP}(D)}{mg} = \frac{c}{D^4} \tag{33}$$

where

$$c = 0.1728\ \mu\text{m}^4$$

for the specific case of ^{40}K atoms and a sapphire surface with $\epsilon_0 = 9.4$ at temperature $T =$ 300 K.

Actually, the same ratio also measures the shift in the length of the interval of oscillation:

$$\frac{\Delta L}{L} \approx \frac{c}{D^4}. \tag{34}$$

Then, in [7] they also performed an accurate numerical analysis for the non-interacting many fermions system taking fully into account the spatial inhomogeneity of the Casimir–Polder force over the atomic cloud; the comparison between these two models is explained in figure 3 by [7].

Here, we perform the same analysis by making use of our analytical formulae (3) and (4).

For the sake of argument we assume that the first energy band is given by $E_1(k) = \frac{1}{2}\delta[1 - \cos(kd)]$. In particular, $\epsilon(x) = V_{\rm CP}(x)/x$ (here we should recall that, for the sake of simplicity, we have placed the surface at $x = 0$ and the endpoints of the interval of oscillation are placed at $x = D - \frac{1}{2}L$ and $x = D + \frac{1}{2}L$) and we obtain that

$$\begin{aligned}\Delta L &= -\frac{1}{F}\int_0^L \epsilon\left(D - \frac{1}{2}L + x\right) \mathrm{d}x = \int_0^L \frac{1}{3}c\left(D - \frac{1}{2}L + x\right)^{-4} \mathrm{d}x \\ &= c\frac{16}{9}\frac{L(L^2 + 12D^2)}{(4D^2 - L^2)^3}.\end{aligned}$$

For what concerns the period T_B^ϵ we have that the shift period is given by

$$\begin{aligned}\Delta T_B &= -\frac{2\hbar}{F}\int_0^L \frac{\epsilon\left(D - \frac{1}{2}L + x\right)}{E_1'\left[E_1^{-1}\left(Fx + E_1^b\right)\right]} \mathrm{d}x \\ &= -\frac{2\hbar}{\mathrm{d}F^2}\int_0^{\delta/F} \frac{\epsilon\left(D - \frac{1}{2}L + x\right)}{\sqrt{x}\sqrt{L - x}} \mathrm{d}x\end{aligned}$$

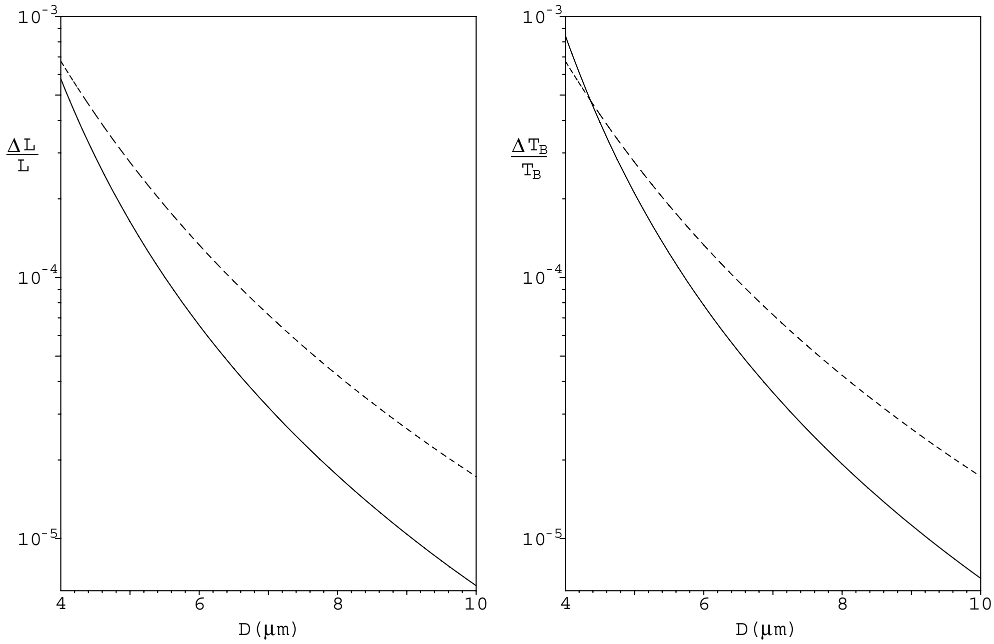

Figure 4. We plot the ratios $\Delta L/L$ (left) and $\Delta T_B/T_B$ (right) as a function of the distance D between the middle of the lattice and the surface. The length L of the periodical lattice is $L = 4\,\mu$m. Dotted lines represent the case (33)–(34) when we assume that the external Casimir–Polder force acts on the atomic cloud as a homogeneous force. Full lines represent the case (35) when we also take into account the effect of the spatial inhomogeneity of the Casimir–Polder force on the atomic cloud.

$$= \frac{T_B c}{3\pi}\int_0^L \frac{1}{\left(D - \frac{1}{2}L + x\right)^4 \sqrt{x}\sqrt{L-x}}\,\mathrm{d}x$$
$$= T_B c \frac{16D(8D^2 + 3L^2)}{3(4D^2 - L^2)^{7/2}}.$$

Hence,

$$\frac{\Delta L}{L} = c\frac{16}{9}\frac{L^2 + 12D^2}{(L^2 - 4D^2)^3} \qquad \text{and} \qquad \frac{\Delta T_B}{T_B} = c\frac{16D(8D^2 + 3L^2)}{3(4D^2 - L^2)^{7/2}}. \qquad (35)$$

The results of such a calculation are shown in figure 4. In particular it appears that, for what concerns the Bloch period shift, our theoretical result qualitativelly agrees with the numerical one obtained in [7], that is when we take into account the effect of the spatial inhomogeneity then the shift length and the shift period exhibit a more strong dependence on the distance between the surface and the atomic cloud.

Acknowledgments

This work is partially supported by the INdAM Project *'Mathematical modeling and numerical analysis of quantum systems with applications to nanosciences'*. The author thanks Professor Vincenzo Grecchi for useful discussions.

Appendix

Here, we collect the proofs of the main results given in section 2 concerning Bloch oscillators.

A.1. Proof of equation (10)

The proof of the acceleration theorem in this form is quite simple. Indeed, let

$$\Phi(k,\tau) = \sum_{n=1}^{\infty} |a_n(k,\tau)|^2,$$

then, keeping in mind that $X_{n,\ell} = \bar{X}_{\ell,n}$, from equation (8) it follows that the function Φ satisfies to the equation

$$\left[F\frac{\partial}{\partial k} + F\frac{\partial}{\partial \tau}\right]\Phi = 0$$

which has a solution of the type $\Phi(k,\tau) = \Phi(k-\tau)$ where Φ is an arbitrary periodic function of its argument satisfying the normalization condition

$$\int_{\mathcal{B}} |\Phi(k)|^2\, \mathrm{d}k = \sum_{n=1}^{\infty}\int_{\mathcal{B}} |a_n(k,\tau)|^2\, \mathrm{d}k = 1.$$

Hence,

$$\begin{aligned}\langle k\rangle^\tau &= \sum_{n=1}^{\infty}\int_{\mathcal{B}} k|a_n(k,\tau)|^2\, \mathrm{d}k = \int_{\mathcal{B}} k\Phi(k-\tau)\, \mathrm{d}k\\ &= \int_{\mathcal{B}} [k+\tau]\Phi(k)\, \mathrm{d}k = \langle k\rangle^0 + \tau.\end{aligned}$$

A.2. Proof of equation (12)

To this end, we set $\xi = k - \tau$, then equation (11) takes the form

$$\left[E_n(k) - FX_{n,n}(k) - \mathrm{i}F\frac{\mathrm{d}}{\mathrm{d}k}\right]a_n = 0, \qquad n = 1, 2, \ldots$$

which has a solution

$$a_n = h_n(\xi)\exp\left[-\frac{\mathrm{i}}{F}\int_0^k [E_n(q) - FX_{n,n}(q)]\, \mathrm{d}q\right]$$

where $h_n(\xi)$ is an arbitrary function of its argument. In order to have a_n^0 at the time $\tau = 0$, that is at $\xi = k$, then

$$h_n(\xi) = a_n^0(\xi)\exp\left[\frac{\mathrm{i}}{F}\int_0^\xi [E_n(q) - FX_{n,n}(q)]\, \mathrm{d}q\right]$$

obtaining (12) and (13).

A.3. Proof of equations (15) and (16)

Here, following [12] we make use of the acceleration theorem for the motion of wave packet in the quasi-momentum representation in the form (12). In fact, a different proof, under some technical assumptions on the external field, has also been given by [14].

From equations (6) and (7), and from the acceleration theorem in the form (12), then formally it follows that

$$\begin{aligned} Fx\psi &= -\mathrm{i}F\frac{\partial\psi}{\partial\tau} + \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right]\psi \\ &= \sum_{n=1}^{\infty}\int_{\mathcal{B}} d_n(k,\tau)\varphi_n(k,x)\,\mathrm{d}k \end{aligned}$$

where

$$\begin{aligned} d_n(k,\tau) &= -\mathrm{i}F\frac{\partial a_n(k,\tau)}{\partial\tau} + E_n(k)a_n(k,\tau) \\ &= \left[\mathrm{i}F\frac{\partial a_n^0(k-\tau)}{\partial k} + (E_n(k) - E_n(k-\tau))\,a_n^0(k-\tau)\right]\mathrm{e}^{\mathrm{i}\theta_n(k,\tau)}. \end{aligned}$$

From this fact and since $\int_{-\infty}^{+\infty}\bar{\varphi}_m(x,k')\varphi_n(x,k)\,\mathrm{d}x = \delta(k-k')\delta_n^m$ it follows that

$$\begin{aligned} F\langle x\rangle^\tau &= \int_{-\infty}^{+\infty}\bar{\psi}(x,\tau)Fx\psi(x,\tau)\,\mathrm{d}x \\ &= \sum_{n,m=1}^{\infty}\int_{-\infty}^{+\infty}\int_{\mathcal{B}}\int_{\mathcal{B}}\bar{a}_m(k',\tau)d_n(k,\tau)\bar{\varphi}_m(k',x)\varphi_n(k,x)\,\mathrm{d}k\,\mathrm{d}k'\,\mathrm{d}x \\ &= \sum_{n=1}^{\infty}\int_{\mathcal{B}}\bar{a}_n(k,\tau)d_n(k,\tau)\,\mathrm{d}k \\ &= g(\tau) + \sum_{n=1}^{\infty}\int_{\mathcal{B}}[E_n(k) - E_n(k-\tau)]\left|a_n^0(k-\tau)\right|^2\mathrm{d}k \\ &= g(0) + \sum_{n=1}^{\infty}\int_{\mathcal{B}}[E_n(k+\tau) - E_n(k)]\left|a_n^0(k)\right|^2\mathrm{d}k \end{aligned}$$

where the term

$$\begin{aligned} g(\tau) &= \sum_{n=1}^{\infty}\mathrm{i}F\int_{\mathcal{B}}\bar{a}_n^0(k-\tau)\frac{\partial a_n^0(k-\tau)}{\partial k}\,\mathrm{d}k \\ &= \sum_{n=1}^{\infty}\mathrm{i}F\int_{\mathcal{B}}\bar{a}_n^0(k)\frac{\partial a_n^0(k)}{\partial k}\,\mathrm{d}k = g(0) \end{aligned}$$

is independent of time and, from the previous equation, it coincides with $F\langle x\rangle^0$. Hence (15) follows.

In order to compute the variance $S^\tau = \langle[x - \langle x\rangle^\tau]^2\rangle^\tau = \langle x^2\rangle^\tau - [\langle x\rangle^\tau]^2$ we make use of the above equation where, assuming that the function a^0 is real-valued, we obtain that

$$\begin{aligned} F^2\langle x^2\rangle^\tau &= \langle Fx\psi, Fx\psi\rangle \\ &= \left\langle\sum_{m=1}^{\infty}\int_{\mathcal{B}} d_m(k',\tau)\varphi_m(k',x)\,\mathrm{d}k', \sum_{n=1}^{\infty}\int_{\mathcal{B}} d_n(k,\tau)\varphi_n(k,x)\,\mathrm{d}k\right\rangle \\ &= \sum_{n=1}^{\infty}\int_{\mathcal{B}}|d_n(k,\tau)|^2\,\mathrm{d}k \\ &= \sum_{n=1}^{\infty}\int_{\mathcal{B}}|d_n(k+\tau,\tau)|^2\,\mathrm{d}k \end{aligned}$$

$$= \sum_{n=1}^{\infty} \int_{\mathcal{B}} \left| \mathrm{i}F \frac{\partial a_n^0(k)}{\partial k} + [E_n(k+\tau) - E_n(k)]\, a_n^0(k) \right|^2 \mathrm{d}k$$

$$= F^2 \langle x^2 \rangle^0 + \sum_{n=1}^{\infty} \int_{\mathcal{B}} [E_n(k+\tau) - E_n(k)]^2 \left| a_n^0(k) \right|^2 \mathrm{d}k$$

where

$$\langle x^2 \rangle^0 = \sum_{n=1}^{\infty} \int_{\mathcal{B}} \left| \frac{\partial a_n^0(k)}{\partial k} \right|^2 \mathrm{d}k$$

proving so equation (16).

References

[1] Allaire G, Palombaro M and Rauch J 2007 Diffractive behavior of the wave equation in periodic media: weak convergence analysis (*Preprint* CMAP 628/2007)

[2] Bentosela F *et al* 1988 Analyticity and asymptotics for the Wannier–Stark states *J. Phys. A: Math. Gen.* **21** 3321

[3] Bloch F 1928 Über die quantenmechanik der elektronen in kristallgittern *Z. Phys.* **52** 555

[4] Bouchard A M and Luban M 1995 Bloch oscillations and other dynamical phenomena of electrons in semiconductor superlattices *Phys. Rev.* B **52** 5105

[5] Callaway J 1974 *Quantum Theory of the Solid State* (New York: Academic)

[6] Carles R, Markowich P A and Sparber C 2004 Semiclassical asymptotics for weakly nonlinear Bloch waves *J. Stat. Phys.* **117** 343

[7] Carusotto I, Pitaevskii L, Stringari S, Modugno G and Inguscio M 2005 Sensitive measurament of forces at the micron scale using Bloch oscillations of ultracold atoms *Phys. Rev. Lett.* **95** 093202

[8] Fallani L, Fort C and Inguscio M 2005 Bose–Einstein condensates in optical potentials *Riv. Nuovo Cimento* **28** 1

[9] Fischer M C, Madison K W, Niu Q and Raizen M G 1998 Observation of Rabi oscillations between Bloch bands in an optical potential *Phys. Rev.* A **58** R2648

[10] Grecchi V and Sacchetti A 1997 Metastable Bloch oscillators *Phys. Rev. Lett.* **78** 4474

[11] Grecchi V and Sacchetti A 1998 Wannier–Bloch oscillators *Commun. Math. Phys.* **197** 553

[12] Grecchi V and Sacchetti A 2001 Acceleration theorem for Bloch oscillators *Phys. Rev.* B **63** 212303

[13] Houston H V 1940 Acceleration of electrons in a crystal lattice *Phys. Rev.* **57** 184

[14] Hövermann F, Spohn H and Teufel S 2001 Semiclassical limit for the Schrödinger equation with a short scale periodic potential *Commun. Math. Phys.* **215** 609

[15] Kittel C 1987 *Quantum Theory of Solids 2nd edn* (New York: Wiley)

[16] Niu Q, Zaho X-G, Georakis G A and Raizen M G 1996 Atomic Landau–Zener Tunneling and Wannier–Stark Ladders in optical potentials *Phys. Rev. Lett.* **76** 4504

[17] Resta R 2000 Manifestations of Berry's phase in molecules and condensed matter *J. Phys.: Condens. Matter* **12** R107

[18] Sandaran G and Niu Q 1999 Wave packet dynamics in slowly perturbed crystals: gradient corrections and Berry-phase effects *Phys. Rev.* B **59** 14915

[19] Wacker A 2002 Semiconductor superlattices: a model system for nonlinear transport *Phys. Rep.* **357** 1

[20] Wannier G H 1962 Dynamics of band electrons in electric and magnetic fields *Rev. Mod. Phys.* **34** 645

[21] Wilkinson S R *et al* 1997 Experimental evidence for non exponential decay in quantum tunneling *Nature* **387** 575

[22] Zak J 1989 Berry's phase for energy bands in solids *Phys. Rev. Lett.* **62** 2747

[23] Zener C 1934 A theory of electronic breakdown of solid dielectrics *Proc. R. Soc.* A **145** 523